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Application of Classical Inequalities

This paper is devoted to the application of classical inequalities in algebraic problems. The theoretical background consists of formulations of various fundamental inequalities such as Cauchy's, Jensen's, Chebyshev's, etc. Application of these methods is included in the second part where some Olympic-style problems are stated and proved. The main goal of the study is to obtain a partial classification of classical inequalities in order to show how their usage leads to certain simplicity in proving inequalities.

1. Introduction

Learning to prove inequalities is not an easy task to achieve. It is, in most cases, a very demanding and grueling work. Tremendous efforts, great experience and a bit of luck are required to tackle even a simple-looking inequality. Unlike most parts of mathematics, there isn't a universal way of dealing with this type of problems.

Yet, a variety of methods exists, without a single clue to point out when and which of them should be used. Furthermore, in each particular case there is a strong probability that failure to use the most suitable method will lead to failure to solve the problem itself. Therefore, a vital criterion is needed to lead the solver to the correct method.

One of the most useful methods for proving inequalities is based on classical inequalities. They are called classical, because they are usually helpful in proving various types of inequalities.

Almost all inequalities stated in the paper are true for any nonnegative real numbers, which is a firm reason for their universal use and application. As a matter of fact, just the definition "classical" is an evidence of their wide applicability.

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2. Theoretical background

2.1. Inequalities between harmonic mean, geometric mean, arithmetic mean and root mean square [2]

As it is well known for the nonnegative numbers $x_1, x_2, \ldots, x_n$ the following inequalities are fulfilled:

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}} \le \sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1+x_2+\ldots+x_n}{n} \le \sqrt{\frac{x_1^2+x_2^2+\ldots+x_n^2}{n}}$$

harmonic mean (HM)	geometric mean (GM)	arithmetic mean (AM)	root mean square (RMS)

Equality holds when $x_1 = x_2 = \ldots = x_n$.

2.2. Cauchy-Schwarz inequality [1]

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers and n is a natural number, then the inequality:

$$(a_1^2 + a_2^2 + \ldots + a_n^2)\,(b_1^2 + b_2^2 + \ldots + b_n^2) \ge (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2$$

is valid. Equality holds when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$, i. e. when $a_1, a_2, \ldots, a_n$ are proportional to $b_1, b_2, \ldots, b_n$, respectively.

2.3. Chebyshev's inequality [7]

If $x_1 \ge x_2 \ge \ldots \ge x_n \ge 0$, $y_1 \ge y_2 \ge \ldots \ge y_n \ge 0$, then:

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \cdot \frac{y_1 + y_2 + \ldots + y_n}{n} \le \frac{x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}{n}$$

holds true.

If one of the above sequences is reversely ordered, then the opposite inequality is valid. Chebyshev's inequality can be generalized for more than two sequences. Equality holds when $x_1 = x_2 = \ldots = x_n$ and $y_1 = y_2 = \ldots = y_n$.

2.4. Young's inequality [7]

If a, b, p and q are nonnegative numbers and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality:

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

is valid. Equality is achieved when $a^p = b^q$.

2.5. Hölder's inequality [1]

For every two sequences of nonnegative reals a_i and b_i $(i = 1, 2, \ldots, n)$,

$$(a_1^p + a_2^p + \ldots + a_n^p)^{1/p} (b_1^q + b_2^q + \ldots + b_n^q)^{1/q} \geq a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

is fulfilled, where p and q satisfy the conditions $\frac{1}{p} + \frac{1}{q} = 1$, p *symbol*>>Č1, q *symbol*>>Č1. Equality holds when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$.

2.6. Minkowski's inequality [1]

$$(a_1^p + a_2^p + \ldots + a_n^p)^{1/p} (b_1^p + b_2^p + \ldots + b_n^p)^{1/p} \geq$$
$$\geq \left((a_1 + b_1)^p + (a_2 + b_2)^p + \ldots + (a_n + b_n)^p\right)^{1/p}$$

where $p \geq 1$. Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$. If $p < 1$, the opposite inequality is valid.

2.7. Jensen's inequality [7]

If $y = f(x)$ is a convex function and $x_1, x_2, \ldots, x_n$ are real numbers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary positive numbers with $\sum_{i=1}^{n} \alpha_i = 1$, then the inequality:

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n)$$

holds true.

Equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Another useful formulation of Jensen's inequality is the following: If $f(x)$ is convex in the interval Δ and $x_1, x_2, \ldots, x_n \in \Delta$, then:

$$f\left(\frac{\sum p_i x_i}{\sum p_i}\right) \leq \frac{\sum p_i f(x_i)}{\sum p_i}, \text{ where } p_1, p_2, \ldots, p_n > 0$$

Theorem: If $f(x)$ is a continuous function in the interval Δ, $f'(x)$ and $f''(x)$ are defined on Δ and $f''(x) \geq 0$, then $f(x)$ is convex.

2.8. One “beautiful” inequality [7]

For the positive reals $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n \neq 0$, the following inequality holds true

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\ldots+a_n)^2}{b_1+b_2+\ldots+b_n}$$

with equality when $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}$.

This inequality is called “beautiful” because it helps very often in proving different types of problems.

Below, an easier proof (cf. [6]) of the “beautiful” inequality is presented using induction.

Proof: When $n=2$, the inequality reduces to:

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2} \geq \frac{(a_1+a_2)^2}{b_1+b_2} \;\Big|\; \cdot b_1 b_2 (b_1+b_2) > 0$$

After multiplying by the positive number $b_1 b_2 (b_1+b_2)$, we obtain consecutively:

$$a_1^2 b_2 (b_1+b_2)+a_2^2 b_1 (b_1+b_2) \geq (a_1+a_2)^2 b_1 b_2 \Leftrightarrow$$
$$\Leftrightarrow a_1^2 b_1 b_2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_1 b_2 \geq$$
$$\geq a_1^2 b_1 b_2 + 2 a_1 a_2 b_1 b_2 + a_2^2 b_1 b_2 \Leftrightarrow$$
$$\Leftrightarrow (a_1 b_2 - a_2 b_1)^2 \geq 0$$

and the last inequality is obviously true.

If the statement is true for $n=k$ then

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_k^2}{b_k}+\frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1+a_2+\ldots+a_k)^2}{b_1+b_2+\ldots+b_k}+\frac{a_{k+1}^2}{b_{k+1}} \geq \frac{(a_1+a_2+\ldots+a_{k+1})^2}{b_1+b_2+\ldots+b_{k+1}}$$

The last one follows after applying the above already proved statement for $n=2$. Thus the inequality holds true for every natural n.

3. Applications

Problem 1 (IMO 2001, South Korea).

Prove that $\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$, where a, b and c are arbitrary positive reals.

Solution 1 (AM-GM inequality) [9].

First, we shall show that:

$$\frac{a}{\sqrt{a^2+8\,b\,c}} \geq \frac{a^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}$$

which is equivalent to:

$$\left(a^{4/3}+b^{4/3}+c^{4/3}\right) \geq a^{2/3}\,(a^2+8bc)$$

From AM-GM inequality it follows

$$\left(a^{4/3}+b^{4/3}+c^{4/3}\right)-\left(a^{4/3}\right)^2 = \left(b^{4/3}+c^{4/3}\right)\left(a^{4/3}+a^{4/3}+b^{4/3}+c^{4/3}\right) \geq$$

$$\geq 2\,b^{2/3}\,c^{2/3}\cdot 4\,a^{2/3}\,c^{1/3}\,b^{1/3} = 8\,a^{2/3}\,bc$$

Hence,

$$\left(a^{4/3}+b^{4/3}+c^{4/3}\right) \geq \left(a^{4/3}\right)^2 + 8\,a^{2/3}bc = a^{2/3}(a^2+8bc)$$

Further, $\frac{a}{\sqrt{a^2+8bc}} \geq \frac{a^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}$, analogously:

$\frac{b}{\sqrt{b^2+8ac}} \geq \frac{b^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}$ and $\frac{c}{\sqrt{c^2+8ab}} \geq \frac{c^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}$ holds true.

Now it is enough to sum up the left and the right-hand sides of the last three inequalities to prove the problem.

Solution 2 (Cauchy-Schwarz inequality) [6].

Let be:

$$x_1 = \frac{\sqrt{a}}{\sqrt[4]{a^2+8bc}},\; y_1 = \sqrt{a}.\sqrt[4]{a^2+8bc};\; x_2 = \frac{\sqrt{b}}{\sqrt[4]{b^2+8ac}},\; y_2 = \sqrt{b}.\sqrt[4]{b^2+8ac};$$

$$x_3 = \frac{\sqrt{c}}{\sqrt[4]{c^2+8ba}},\; y_3 = \sqrt{c}.\sqrt[4]{c^2+8ba}.$$

Then using Cauchy-Schwarz inequality it follows:

$$(a+b+c)^2 \leq \left(\frac{a}{\sqrt{a^2+8\,b\,c}}+\frac{b}{\sqrt{b^2+8\,c\,a}}+\frac{c}{\sqrt{c^2+8\,a\,b}}\right)\cdot$$

$$\cdot\,(a\sqrt{a^2+8bc}+b\sqrt{b^2+8ca}+c\sqrt{c^2+8ab}) \quad (1)$$

Using again the Cauchy-Schwarz inequality, but now within the numbers:

$$x_1 = \sqrt{a},\; y_1 = \sqrt{a\,(a^2+8bc)};\quad x_2 = \sqrt{b},\; y_2 = \sqrt{b\,(b^2+8ca)};$$

$$x_3 = \sqrt{c},\; y_3 = \sqrt{c\,(c^2+8ab)}.$$

It follows that:

$$\left(a\sqrt{\left(a^2+8bc\right)}+b\sqrt{\left(b^2+8ac\right)}+c\sqrt{\left(c^2+8ab\right)}\right)^2 \le \left(a+b+c\right)\left(a^3+b^3+c^3+24abc\right). \quad (2)$$

Therefore the problem is reduced to proving the inequality:

$$a^3+b^3+c^3+24abc \le (a+b+c)^3 \qquad (3)$$

which is equivalent to $\dfrac{a^2b+b^2c+c^2a+a^2c+b^2a+c^2b}{6} \ge abc$.

The last inequality follows by direct application of AM-GM inequality. Thus (3) is true.

Hence, from (1), (2) and (3) we conclude that

$$\left(a+b+c\right)^2 \le \left(\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\right).(a\sqrt{a^2+8bc}+b\sqrt{b^2+8ca}+c\sqrt{c^2+8ab}) \le$$

$$\le \left(\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\right)\left(a+b+c\right)^2$$

Therefore, $\dfrac{a}{\sqrt{a^2+8\,b\,c}}+\dfrac{b}{\sqrt{b^2+8\,c\,a}}+\dfrac{c}{\sqrt{c^2+8\,a\,b}} \ge 1$ is satisfied.

Equality holds when $a^2+8bc=b^2+8ca=c^2+8ab$, $a^2b=b^2c=c^2a=$ $=a^2c=b^2a=c^2b$ i. e. for $a=b=c$.

Problem 2.

Prove the inequality

$$a_1+a_2+\ldots+a_n-n\sqrt[n]{a_1a_2\ldots a_n} \ge \left(\sqrt{a_1}-\sqrt{a_2}\right)^2$$

where $a_1, a_2, \ldots, a_n$ are positive numbers.

Solution 1 (AM-GM inequality).

Let $A_k=(a_1+a_2+\ldots+a_k)/k$, $G_k=\sqrt[k]{a_1a_2\ldots a_k}$, where $k=2, \ldots, n$.

First, it will be proved that the following remarkable inequality is valid:

$$(k+1)\,(A_{k+1}-G_{k+1}) \ge k\,(A_k-G_k) \qquad (4)$$

Denote $x=\sqrt[k+1]{a_{k+1}}$, $g=\sqrt[k+1]{G_k}$. It is easily seen that:

$$A_{k+1}=\frac{kA_k+x^{k+1}}{k+1}, G_{k+1}=\sqrt[k+1]{G_k{}^k x^{k+1}}=g^k x$$

Therefore (4) can be expanded to $x^{k+1} - (k+1)\, g^k x + kg^{k+1} \geq 0$. The latter can be easily proved as it is known that the left-hand side is divisible by $(x-g)^2$:

$$x\,(x^k - g^k) - k\,g^k(x-g) = (x-g)\,(x^k + x^{k-1}g + \ldots + xg^{k-1} - kg^k) =$$
$$= (x-g)^2\,(x^{k-1} + 2x^{k-2}g + 3x^{k-3}g^2 + \ldots + (k-1)\,xg^k + kg^k)$$

Thus, the left-hand side is always positive and (4) is proved. Hence,

$$n\,(A_n - G_n) \geq (n-1)\,(A_{n-1} - G_{n-1}) \geq \ldots \geq 2(A_2 - G_2) =$$
$$= a_1 + a_2 - 2\sqrt{a_1 a_2} = \left(\sqrt{a_2} - \sqrt{a_1}\right)^2$$

Remark 1. This proof can be directly applied to obtain the AM-GM inequality.

Solution 2. Rewriting the inequality in the form

$$\sqrt{a_1 a_2} + \sqrt{a_1 a_2} + a_3 + \ldots + a_n \geq n\sqrt[n]{a_1 a_2 \ldots a_n}$$

and using the AM-GM inequality for the numbers $\sqrt{a_1\, a_2}$, $\sqrt{a_1\, a_2}$, a_3, … …, a_n proves the problem.

Remark 2. Each of the above two solutions shows that equality is obtained iff $a_3 = a_4 = \ldots = a_n = \sqrt{a_1\, a_2}$.

Problem 3. (IMO 1987, Cuba) [8].

Let $x_1, x_2, \ldots, x_n$ be reals, such that $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$ is valid. Prove that for every natural number $k \geq 2$, there exist integers a_1, a_2, … a_n, $a_1^2 + a_2^2 + \ldots + a_n \neq 0$, such that $|a_i| \geq k-1$ for $i = 1, 2, \ldots, n$ and $|a_1 x_1 + a_2\, x_2 + \ldots + a_n\, x_n| \leq \dfrac{(k-1)\sqrt{n}}{k^n - 1}$.

Proof: Applying Cauchy-Schwarz inequality, we obtain:

$$(a_1 x_1) + (a_2 x_2) + \ldots + (a_n x_n) \leq \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \leq (k-1)\sqrt{n},$$

where $a_i \in \{0, 1, \ldots, n\}$ for $i = 1, 2, \ldots, n$. The number of sums of the form $\sum_{i=1}^{n} |a_i x_i|$ for different values of a_i is k^n. They are all in the interval $[0, (k-1)\sqrt{n}]$, which length is $(k-1)\sqrt{n}$.

If the interval is divided into $k^n - 1$ equal parts, at least one of them will contain two sums $\sum_{i=1}^{n} |a_i x_i|$ and $\sum_{i=1}^{n} |b_i x_i|$, thus:

$$0 < \left| \sum_{i=1}^{n} |b_i x_i| - \sum_{i=1}^{n} |a_i x_i| \right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1} \Rightarrow$$

$$\Rightarrow 0 < \left| \sum_{i=1}^{n} (|b_i| - |a_i|)\, |x_i| \right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$$

Let's define the numbers c_i, $i = 1, 2, \ldots, n$, in the following way:

$$c_i = \begin{cases} |b_i| - |a_i|, & \text{if } x_i \geq 0 \\ |a_i| - |b_i|, & \text{if } x_i < 0 \end{cases}$$

Then c_i $(i = 1, 2, \ldots, n)$ are integers, not all equal to 0 and $|c_i| \leq k - 1$. Also, $c_i x_i = (|b_i| - |a_i|)|x_i|$ holds true $\Rightarrow$

$$\Rightarrow \text{the inequality } \left| \sum_{i=1}^{n} x_i c_i \right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1} \text{ is proven.}$$

Problem 4 (IMO 1979, England) [6].

Find all reals a, for which there exist nonnegative numbers x_1, x_2, x_3, x_4, x_5, such that:

$$\begin{vmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = a \\ x_1 + 2^3 x_2 + 3^3 x_3 + 4^3 x_4 + 5^3 x_5 = a^2 \\ x_1 + 2^5 x_2 + 3^5 x_3 + 4^5 x_4 + 5^5 x_5 = a^3 \end{vmatrix}$$

Solution: Applying Cauchy-Schwarz inequality for the numbers $a_1 = \sqrt{x_1}$, $b_1 = \sqrt{x_1}$; $a_2 = \sqrt{2x_2}$, $b_2 = \sqrt{2^5 x_2}$; $a_3 = \sqrt{3x_3}$, $b_3 = \sqrt{3^5 x_3}$; $a_4 = \sqrt{4x_4}$, $b_4 = \sqrt{4^5 x_4}$; $a_5 = \sqrt{5x_5}$, $b_5 = \sqrt{5^5 x_5}$, $n = 5$, follows that:

$$(\sqrt{x_1} \cdot \sqrt{x_1} + \sqrt{2x_2} \cdot \sqrt{2^5 x_2} + \ldots + \sqrt{5x_5} \cdot \sqrt{5^5 x_5})^2 \leq$$
$$\leq (x_1 + 2x_2 + \ldots + 5x_5)(x_1 + 2^5 x_2 + \ldots + 5^5 x_5)$$

The above inequality is reduced to $\left(a^2\right)^2 \le a \cdot a^3$, which is in fact an equality.

Equality holds if only if there is equality in Cauchy-Schwarz inequality, i. e. when $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\frac{a_4}{b_4}=\frac{a_5}{b_5}=x$, or equivalently:

$$\sqrt{x_1}=x\sqrt{x_1} \Leftrightarrow (1-x)\sqrt{x_1}=0;\ \sqrt{2x_2}=x\sqrt{2^5x_2} \Leftrightarrow (1-2^2x)\sqrt{x_2}=0$$

$$\sqrt{3x_3}=x\sqrt{3^5x_3} \Leftrightarrow (1-3^2x)\sqrt{x_3}=0$$

$$\sqrt{4x_4}=x\sqrt{4^5x_4} \Leftrightarrow (1-4^2x)\sqrt{x_4}=0$$

$$\sqrt{5x_5}=x\sqrt{5^5x_5} \Leftrightarrow (1-5^2x)\sqrt{x_5}=0$$

If $x=1$, $x_1=x_2=x_3=x_4=x_5=0$ and $a=0$ or $x_2=x_3=x_4=x_5=0$ and $x_1=a=1$, then the solutions are (0,0,0,0,0) and (1,0,0,0,0).

If $x=\frac{1}{4}$, $x_1=x_3=x_4=x_5=0$ and $a=4$, then the solution is (0,2,0,0,0).

If $x=\frac{1}{9}$, $x_1=x_2=x_4=x_5=0 \Rightarrow a=9, x_3=3$ and the solution is (0,0,3,0,0).

If $x=\frac{1}{16}$, then $a=16$, $x_4=4$ and (0,0,0,4,0) is a solution.

If $x=\frac{1}{25}$, then $a=25$, $x_5=5$ and the solution is (0,0,0,0,5).

Problem 5 [4].

Let x,y,z,p,q,r be positive reals, such that $x^p y^q z^r=1$ and $p+q+r=1$. Prove the inequality:

$$\frac{p^2x^2}{qy+rz}+\frac{q^2y^2}{px+rz}+\frac{r^2z^2}{px+qy} \ge \frac{1}{2}$$

Proof: First, the Young's inequality and its generalization will be proved, because it will be necessary to solve the problem. The steps of the proof are separated in three lemmas:

Lemma 1. $x^\alpha-\alpha x\le 1-\alpha$, where $x>0$, $0<\alpha<1$.

Proof: For $x>0$ consider the function $f(x)=x^\alpha-\alpha x$, where $0<\alpha<1$. We easily get that:

$$f'(x)=\alpha\left(x^{\alpha-1}-1\right)\begin{cases} >0 & \text{when } 0<x<1 \\ \le 0 & \text{when } x\ge 1\end{cases}$$

Therefore the function increases, when x varies in the interval (0; 1] and decreases in the interval [1; +∞). Now it is clear that $f(1)=1-\alpha$ is the greatest value of the function in the interval (0, + ∞) $\Rightarrow x^{\alpha} - \alpha x \leq$. $\leq 1 - \alpha$.

Lemma 2. $a^{\alpha} b^{\beta} \leq \alpha a + \beta b$ holds true where a, b, α, $\beta > 0$ and $\alpha + \beta = 1$.

Substituting $x = \frac{a}{b}$ in Lemma 1 and using $\alpha + \beta = 1$, i. e. $\beta = 1 - \alpha$ it follows:

$$\frac{a^{\alpha}}{b^{\alpha}} - \alpha \cdot \frac{a}{b} \leq \beta \;\Big|\; \cdot b \Rightarrow a^{\alpha} b^{1-\alpha} \leq \beta b + \alpha a \Rightarrow$$

$$\Rightarrow a^{\alpha} b^{\beta} \leq \alpha a + \beta b$$

Lemma 3. $a^{\alpha} b^{\beta} c^{\gamma} \leq \alpha a + \beta b + \gamma c$ is valid, where a, b, c, α, β, $\gamma > 0$ and $\alpha + \beta + \gamma = 1$.

Proof: Applying twice *Lemma 2* implies

$$a^{\alpha} b^{\beta} c^{\gamma} = a^{\alpha} \left(b^{\frac{\beta}{\beta+\gamma}} c^{\frac{\gamma}{\beta+\gamma}} \right)^{\beta+\gamma} \leq \alpha a + (\beta + \gamma)\, b^{\frac{\beta}{\beta+\gamma}} c^{\frac{\gamma}{\beta+\gamma}} \leq$$

$$\leq \alpha a + (\beta + \gamma) \left(\frac{\beta}{\beta+\gamma} b + \frac{\gamma}{\beta+\gamma} c \right) = \alpha a + \beta b + \gamma c$$

Lemma 3 can be generalized in the following way:

$$a_1^{q_1} a_2^{q_1} \ldots a_n^{q_n} \leq q_1 a_1 + q_2 a_2 + \ldots + q_n a_n$$

where $a_1, \ldots, a_n, q_1, \ldots, q_n > 0$, $q_1 + q_2 + \ldots + q_n = 1$.

Now we go back to the solution of Problem 5. Applying Cauchy-Schwarz inequality for the numbers:

$$a_1 = \frac{p\,x}{\sqrt{q\,y + r\,z}},\; b_1 = \frac{p\,y}{\sqrt{p\,x + r\,z}},\; c_1 = \frac{r\,z}{\sqrt{p\,x + q\,y}},$$

$$a_2 = \sqrt{q\,y + r\,z},\; b_2 = \sqrt{p\,x + r\,z},\; c_2 = \sqrt{p\,x + q\,y}$$

leads to:

$$(p\,x + q\,y + r\,z)^2 \leq \left(\frac{p^2 x^2}{q\,y + r\,z} + \frac{q^2 y^2}{p\,x + r\,z} + \frac{r^2 z^2}{p\,x + q\,y} \right) \cdot 2\,(p\,x + q\,y + r\,z)$$

Hence,

$$\frac{p^2 x^2}{q\,y + r\,z} + \frac{q^2 y^2}{p\,x + r\,z} + \frac{r^2 z^2}{p\,x + q\,y} \geq \frac{1}{2}(p\,x + q\,y + r\,z) \geq$$

$$\geq \frac{x^p y^q z^r}{2} = \frac{1}{2}$$

where the last inequality is obtained by means of Lemma 3. Since $p+q+$ $+r=1$, Lemma 3 can be applied again (with $p=q=r=\frac{1}{3}$, $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$) to imply:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

where a, b, $c>0$ and $abc=1$.

The last inequality is a problem from IMO 1995. In what follows, an elegant proof of that problem will be proposed by the author.

Problem 6 (IMO 1995).

Let a, b, c be positive reals such that $abc=1$. Prove the inequality:

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

Proof: Denote $x=\frac{1}{a}$, $y=\frac{1}{b}$, $z=\frac{1}{c}$. Then x, y, z are positive and $xyz=1$ is valid. Thus, the given inequality is equivalent to:

$$S=\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2} \tag{5}$$

The last inequality can be proved in many different ways. Three of them are given below.

No1: Let $S_1=\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}$. Assume that $x\geq y\geq z$; then

$$\frac{1}{y+z}\geq\frac{1}{z+x}\geq\frac{1}{x+y}$$

Appling Chebyshev's inequality for the triples (x, y, z) and:

$$\left(\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}\right)\Rightarrow$$

$$\Rightarrow S_1=\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}\geq\frac{1}{3}(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)$$

Substituting $u=x+y$, $v=y+z$, $w=z+x$ and applying AM-HM inequality for the numbers u, v, w, it follows that:

$$\frac{1}{3}(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)=\frac{1}{3}\cdot\frac{u+v+w}{2}\cdot\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)\geq\frac{3}{2}$$

$$\Rightarrow S_1=\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}\geq\frac{3}{2} \tag{6}$$

We will prove one generalization to the inequality (6).

Generalization: For every $\alpha \geq 1$ and $x > 0$, $y > 0$, $z > 0$, $xyz = 1$, let:

$$S_\alpha = \frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2} \qquad (7)$$

Applying the Chebyshev's inequality for the triples:

$\left(\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}\right)$ and $(x^{\alpha-1}, y^{\alpha-1}, z^{\alpha-1})$

and then using (6) and AM-GM inequality for the positive numbers $x^{\alpha-1}, y^{\alpha-1}, z^{\alpha-1}$, we get:

$$S_\alpha \geq S_1 \frac{x^{\alpha-1} + y^{\alpha-1} + z^{\alpha-1}}{3} \geq \frac{3}{2}(xyz)^{\frac{\alpha-1}{3}} = \frac{3}{2}$$

Thus, the statement is true for $\alpha = 2 \Rightarrow$ (5) is fulfilled. Equality holds if $\alpha \geq 1$ and $\alpha \leq -2$. It can be easily seen that if we choose $x = \frac{1}{n}$, $y = 1$, $z = 1$ or $x = \frac{1}{n}$, $y = \frac{1}{n}$, $z = n^2$ and n approaches infinity, the inequality is no more valid.

No2 [8]: Using the AM-GM inequality, we obtain consecutively

$[(y+z) + (z+x) + (x+y)] \cdot S =$

$$= x^2 + y^2 + z^2 + (x^2 \frac{z+x}{y+z} + y^2 \frac{y+z}{z+x}) + (y^2 \frac{x+y}{z+x} + z^2 \frac{z+x}{y+z}) +$$

$$+ (z^2 \frac{y+z}{x+y} + x^2 \frac{x+y}{y+z}) \geq x^2 + y^2 + z^2 + 2xy + 2yz + 2xz = (x+y+z)^2 \Rightarrow$$

$$\Rightarrow 2(x+y+z) \cdot S \geq (x+y+z)^2 \Rightarrow$$

$$\Rightarrow S \geq \frac{x+y+z}{2} = \frac{x+y+z}{3} \cdot \frac{3}{2} \geq (xyz)^{1/3} \cdot \frac{3}{2} = \frac{3}{2}$$

No3: Applying Cauchy-Schwarz inequality for the triples:

$\left(a_1 = \frac{x}{\sqrt{y+z}}, a_2 = \frac{y}{\sqrt{z+x}}, a_n = \frac{z}{\sqrt{x+y}}\right)$ and

$(b_1 = \sqrt{y+z}, b_2 = \sqrt{z+x}, b_n = \sqrt{x+y}\ ) \Rightarrow$

$$\Rightarrow (x+y+z)^2 \leq \left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right)(y+z+z+x+x+y) \Rightarrow$$

$\Rightarrow (x+y+z)^2 \leq S \cdot 2\,(x+y+z)$. By means of AM-GM inequality,

$S \geq \frac{x+y+z}{2} = \frac{x+y+z}{3} \cdot \frac{3}{2} \geq (xyz)^{1/3} \cdot \frac{3}{2} = \frac{3}{2}$, and the statement is proved.

Problem 7 (BMO 1984).

Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be positive reals with sum 1. Prove that:

$$\frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \ldots + \frac{x_n}{2-x_n} \geq \frac{n}{2n-1}$$

Solution 1 (the "beautiful" inequality): Since the inequality is symmetric in all cases, it can be assumed that $x_1 \leq x_2 \leq \ldots \leq x_n$, therefore:

$$\frac{1}{2-x_1} \leq \frac{1}{2-x_2} \leq \ldots \leq \frac{1}{2-x_n}$$

Applying Chebyshev's inequality leads to:

$$\frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \ldots + \frac{x_n}{2-x_n} \geq$$

$$\geq \frac{x_1 + x_2 + \ldots + x_n}{n} \cdot \left(\frac{1}{2-x_1} + \frac{1}{2-x_2} + \ldots + \frac{1}{2-x_n} \right) =$$

$$= \frac{1}{n} \left(\frac{1}{2-x_1} + \frac{1}{2-x_2} + \ldots + \frac{1}{2-x_n} \right)$$

From the "beautiful" inequality for the numbers 1, 1, …, 1 and $2-x_1, 2-x_2, \ldots, 2-x_n$:

$$\frac{1}{2-x_1} + \frac{1}{2-x_2} + \ldots + \frac{1}{2-x_n} \geq \frac{n^2}{2-x_1+2-x_2+\ldots+2-x_n} = \frac{n^2}{2n-1} \Rightarrow$$

$$\Rightarrow \frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \ldots + \frac{x_n}{2-x_n} \geq \frac{1}{n} \cdot \frac{n^2}{2n-1} = \frac{n}{2n-1}$$ holds true.

Solution 2 (Cauchy-Schwarz inequality) [6]: Let:

$$a_1 = \sqrt{\frac{x_1}{2-x_1}}, b_1 = \sqrt{x_1(2-x_1)}, \ldots, a_n = \sqrt{\frac{x_n}{2-x_n}}, b_n = \sqrt{x_n(2-x_n)}, n \geq 2$$

The square roots are positive, because $0 < x_1 < 1$, $0 < x_2 < 1$, …, $0 < x_n < 1$. Then from the Cauchy-Schwarz inequality it follows:

$$\frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \ldots + \frac{x_n}{2-x_n} \geq$$

$$\geq \frac{(x_1+x_2+\ldots+x_n)^2}{2(x_1+x_2+\ldots+x_n) - (x_1^2+x_2^2+\ldots+x_n^2)} = \frac{1}{2-(x_1^2+x_2^2+\ldots+x_n^2)}$$

Applying again Cauchy-Schwarz inequality leads to:

$$(x_1 \cdot 1 + x_2 \cdot 1 + \ldots + x_n \cdot 1)^2 \leq (1^2 + 1^2 + \ldots + 1^2)\,(x_1^2 + x_2^2 + \ldots + x_n^2)$$

i. e. $x_1^2 + x_2^2 + \ldots + x_n^2 \geq \frac{1}{n}$, using the fact that $x_1 + x_2 + \ldots + x_n = 1$.

Further,

$$\Rightarrow \frac{1}{2-(x_1^2 + x_2^2 + \ldots + x_n^2)} \geq \frac{1}{2-\frac{1}{n}} = \frac{n}{2n-1} \Rightarrow$$

$$\Rightarrow \frac{x_1}{2-x_1} + \frac{x_2}{2-x_2} + \ldots + \frac{x_n}{2-x_n} \geq \frac{1}{2-(x_1^2 + x_2^2 + \ldots + x_n^2)} \geq \frac{n}{2n-1}$$

Equality is achieved when $x_1 = x_2 = \ldots = x_n = \frac{1}{n}$.

Problem 8 [3].

If a, b, c are positive reals, prove the inequality:

$$\sqrt{a^2 + ab + b^2} + \sqrt{b^2 + bc + c^2} + \sqrt{c^2 + ca + a^2} \geq 3\sqrt{ab + bc + ca}$$

Proof (Hölder's inequality): First, the following inequality will be proved:

$$(a^2 + ab + b^2)\,(b^2 + bc + c^2)\,(c^2 + ca + a^2) \geq (ab + bc + c\,a)^3 \qquad (8)$$

For that purpose, applying Hölder's inequality leads to:

$$ab + bc + ca = (ab)^{\frac{1}{3}}(b^2)^{\frac{1}{3}}(a^2)^{\frac{1}{3}} + (b^2)^{\frac{1}{3}}(bc)^{\frac{1}{3}}(c^2)^{\frac{1}{3}} + (a^2)^{\frac{1}{3}}(c^2)^{\frac{1}{3}}(ac)^{\frac{1}{3}} \leq$$

$$\leq (ab + b^2 + a^2)^{\frac{1}{3}}(b^2 + bc + c^2)^{\frac{1}{3}}(a^2 + c^2 + ac)^{\frac{1}{3}} \Rightarrow$$

$$\Rightarrow (ab + bc + ca)^3 \leq (ab + b^2 + a^2)\,(b^2 + bc + c^2)\,(a^2 + c^2 + ac)$$

Let $ab + bc + ca = D$, $a^2 + ab + b^2 = A$, $b^2 + bc + c^2 = B$, $c^2 + ca + a^2 = C$. Then the inequality in the problem can be written in the form $\sqrt{A} + \sqrt{B} + \sqrt{C} \geq 3\sqrt{D}$ and (8) in the form $ABC \geq D^3$. From AM-GM inequality:

$$\frac{\sqrt{A} + \sqrt{B} + \sqrt{C}}{3} \geq (\sqrt{A}\,\sqrt{B}\,\sqrt{C})^{1/3} = (ABC)^{1/6} \geq (D^3)^{1/6} \Rightarrow$$

$$\Rightarrow \sqrt{A} + \sqrt{B} + \sqrt{C} \geq 3\sqrt{D}$$

Problem 9 (Romanian National Olympiad, 2002).

If a, b and c are nonnegative reals and $a^2 + b^2 + c^2 = 1$, prove that:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{4}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2$$

Proof (the "beautiful" inequality): Applying the "beautiful" inequality for the numbers:

$a_1 = a\sqrt{a}$, $a_2 = b\sqrt{b}$, $a_3 = c\sqrt{c}$, $b_1 = a^2 b^2 + a^2$, $b_2 = b^2 c^2 + b^2$, $b_3 = c^2 a^2 + c^2$ when $n = 3$, we obtain:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} = \frac{(a\sqrt{a})^2}{a^2 b^2 + a^2} + \frac{(b\sqrt{b})^2}{b^2 c^2 + b^2} + \frac{(c\sqrt{c})^2}{c^2 a^2 + c^2} \geq$$

$$\geq \frac{(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2}{a^2 b^2 + b^2 c^2 + c^2 a^2 + 1} \geq$$

$$\geq \frac{1}{\frac{1}{3}+1}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2 = \frac{3}{4}(a\sqrt{a} + b\sqrt{b} + c\sqrt{c})^2$$

But the last inequality is obvious, because:

$1 = (a^2 + b^2 + c^2)^2 \geq 3\,(a^2 b^2 + b^2 c^2 + c^2 a^2)$ (which is equivalent to $(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0$)

$$\Rightarrow \frac{1}{a^2 b^2 + b^2 c^2 + c^2 a^2} \geq \frac{1}{\frac{1}{3}} \Rightarrow \frac{1}{a^2 b^2 + b^2 c^2 + c^2 a^2 + 1} \geq \frac{1}{\frac{1}{3}+1} = \frac{3}{4}$$

Problem 10 [5].

Let $n > 1$ be an integer and $x_1, x_2, \ldots, x_n$ are arbitrary positive numbers such that $x_1^2 + x_2^2 + \ldots + x_n^2 = n$. For arbitrary $p \geq q > 1$, prove that

$$\frac{(x_1 + x_2 + \ldots + x_n)^q + x_1^q}{(x_1 + x_2 + \ldots + x_n)^p - x_1^p} + \ldots + \frac{(x_1 + x_2 + \ldots + x_n)^q + x_n^q}{(x_1 + x_2 + \ldots + x_n)^p - x_n^p} \geq \frac{n\,(n^q + 1)}{n^p - 1}.$$

Proof (Jensen's inequality): Let $s = \sum_{i=1}^{n} x_i$ and $f(x) = \frac{s^q + x^q}{s^p - x^p}$, $x \in (0, s)$. Then the inequality can be written in the form:

$$\sum_{i=1}^{n} f(x_i) \geq \frac{n\,(n^q + 1)}{n^p - 1} \tag{11}$$

Calculating $f''(x)$, it is easy to see that $f''(x) > 0$. Therefore, the function f is convex, so the Jensen's inequality can be applied:

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) = f\left(\frac{s}{n}\right) = \frac{s^q + \left(\frac{s}{n}\right)^q}{s^p - \left(\frac{s}{n}\right)^p} = \frac{n^{p-q}\left(n^q + 1\right)}{s^{p-q}\left(n^p - 1\right)}$$

On the other hand, $\sum_{i=1}^{n} x_i^2 = n$ and from AM-RM inequality it follows that:

$$\frac{s}{n} = \frac{x_1 + x_2 + \ldots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} = 1 \Rightarrow s \leq n$$

Then $\frac{n^{p-q}}{s^{p-q}} \geq 1$ holds true and it follows:

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \geq \frac{(n^q + 1)\, n^{p-q}}{(n^p - 1)\, s^{p-q}} \geq \frac{n^q + 1}{n^p - 1} \Leftrightarrow \sum_{i=1}^{n} f(x_i) \geq \frac{n\,(n^q - 1)}{n^p - 1}$$

4. Conclusion

The classical inequalities are a highly efficient way of solving or proving various types of inequalities. I have tried to show their wide applicability, which in most cases simplifies to a great extent the proof of certain problems. Clear and short solutions can be easily obtained, emphasizing on the fact that the classical inequalities are a very helpful method for solving many algebraic or geometric problems.

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